Dimension of a random Cantor sets

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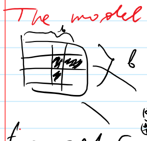
Let us apply the potential theory to prove the following result about random Cantor sets. $b \in \mathbb{N}$, $b \geq 2$.

The model:



b^d cubes in $\mathbb{R}^d$ in Q_0 - unit cube. Keep each with probability p, discard otherwise. Repeat in each cube we kept to to get C_2. $+$ $C = \bigcap C_n$.

Thm $p \leq b^{-d} \Rightarrow C = \emptyset$ a.s.

$p > b^{-d} \Rightarrow C$ is a.s. either empty or $\operatorname{Hdim} C = \operatorname{Mdim} C = d + \log_b p$. The latter occurs with positive probability.

Upper bound

Lemma (Easy bound on Minkowski) Let K be a random set and $\overline{\lim} \frac{\log \mathbb{E}(N(K,\varepsilon))}{|\log \varepsilon|} < \alpha$, $\alpha > 0$. Then a.s. $\overline{\operatorname{Mdim}}\, K \leq \alpha$.

Pf. Take $\alpha_1 > \alpha_2 > \alpha$. Then, for small ε,

$E(N(K, 2^{-n})) \leq 2^{-n\alpha_2}$. So $\mathbb{P}(N(K, 2^{-n}) \geq 2^{n\alpha_1}) \leq 2^{-n\alpha_1} E(N(K, 2^{-n}) \mid N(K, 2^{-n}) \geq 2^{n\alpha_1}) \leq$

$2^{-n(\alpha_1 - \alpha_2)}$ so, by Borel-Cantelli, a.s. $\overline{\lim} \frac{N(K, 2^{-n})}{n \log 2} \leq \alpha_1$ ∎

Return to random Cantor sets.

For any kept cube, let

$q_k = \binom{b^d}{k} p^k (1-p)^{b^d - k}$ be the probability that we kept exactly k cubes inside it. Expected number of the first level kept subcubes is

$m := p b^d = \sum_{k=0}^{b^d} k q_k$.

Let $Z_n(Q)$ is the number (random) of subcubes of b-adic Q kept after n steps. Then, by induction on n,

$\mathbb{E}(Z_n(Q) \mid Q \text{ is kept}) = m^n$.

In particular, $E(N(C, \sqrt{d}\, b^{-n})) \leq m^n$. So by Lemma,

$\overline{\operatorname{Mdim}}\, C \leq \max(\log_b m, 0) = \max(d + \log_b p, 0)$ a.s.

Lower bound.

One can proceed by using the obvious measure on C (define inductively on C_n, pass to the limit) and use Mass Distribution Principle. But we are talking about something happening locally w.r.t. random measure. Too complicated. Instead, let us use the energy of this measure.

Lemma (b-adic friendly formula for capacity). ($\alpha > 0$)

For $K \subset Q_0$, $I_\alpha(\mu) \asymp \sum_{n=0}^{\infty} \left[b^{n\alpha} \left(\sum_{Q - b^{-n} \text{ cubes}} \mu(Q)^2 \right) \right]$

Pf. On one hand

$I_\alpha(\mu) \geq \sum_{n=0}^{\infty} k_\alpha(b^{-n}) (\mu \times \mu)\{(x,y) : b^{-n-1} < |x-y| \leq b^{-n}\} \underset{\text{by parts}}{=}$

$\sum_{n=0}^{\infty} \left(k_\alpha(b^{-n}) - k_\alpha(b^{-n+1})\right) \mu \times \mu\{|x-y| \leq b^{-n}\} \gtrsim$

$\sum_{n=0}^{\infty} b^{n\alpha} \sum_{Q - b^{-(n-\ell)} \text{-cube}} \mu(Q)^2$, where $b^\ell \geq \sqrt{d}$.

On the other hand,

$I_\alpha(\mu) \leq \sum_{n=0}^{\infty} k_\alpha(b^{-n}) \mu \times \mu \{(x,y) : b^{-n} \leq |x-y| < b^{-n+1}\} \underset{\text{sum by parts}}{\leq}$

$\sum_{n=1} b^{n\alpha} \mu \times \mu\{(x,y) : |x-y| \leq b^{1-n}\}$.

We say that b^{-n} cube Q_1 is adjacent to Q_2 if they are neighbors. Notation: $Q_1 \sim Q_2$.

Then

$\mu \times \mu\{(x,y) : |x-y| \leq b^{1-n}\} \leq \sum_{\substack{Q_1 \sim Q_2 \\ Q_1, Q_2 - b^{1-n} \text{ cubes}}} \mu(Q_1)\mu(Q_2)$.

Now, observe first that

$\mu(Q_1)\mu(Q_2) \leq \frac{\mu(Q_1)^2 + \mu(Q_2)^2}{2}$, and second, that

$\#\{Q_1 : Q_1 \sim Q\} = 3^d$.

So, we have

$\mu \times \mu\{(x,y) : b^{-n} < |x-y| \leq b^{1-n}\} \leq 3^d \sum_{Q - b^{1-n} \text{ cube}} \mu(Q)^2$ ∎

Let us return to the random set C.

Let $Z_n(Q)$ be the number of b^{-n} cubes in C which are descendants of Q. $Z_n := Z_n(Q_0)$.

Then $E Z_n = m^n$, and for Q being a b^{-k}-cube, ($k < n$) $Z_n(Q)$ has the same distribution as Z_{n-k}.

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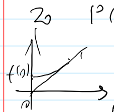
Define now a (random) measure μ on C by $\mu(Q) = \lim_{n\to\infty} m^{-n} Z_n(Q)$.
It is indeed a measure, since $\sum_{\substack{Q \text{ in } C \\ Q \subset Q_1 \\ Q - b^{-k-1}\text{-cube}}} \mu(Q) = \mu(Q_1)$ for any b^{-k}-cube Q, (since it was true for any $n > k$ and $m^{-n} Z_n(Q)$)

But why does the limit exist?

Lemma. $m^{-n} Z_n \to \mu(Q_0)$ a.s. and in L^2 for $m > 1$.
The sets $\{\omega : \mu(Q_0) = 0\}$ and $\{\omega : C = \emptyset\}$
are the same up to a set of probability 0.
When $m \le 1$, these sets are of full probability
When $m > 1$, these sets are of $p < 1$.

Assume lemma, then
$I_\alpha(\mu) \asymp \sum_{n=0}^{\infty} b^{-n(\alpha+1)} \sum_{Q - b^{-n}\text{-cube}} \mu(Q)^2$. Observe that
$\mu(Q)$ has the same distribution as $m^{-n}\mu(Q_0)$, so
$E I_\alpha(\mu) = \sum_{n=0}^{\infty} b^{\alpha n} \cdot m^{-2n} E(\mu(Q_0)^2) P(Q \in C_n) =$
The last probability is exactly p^n (need to make the pick of Q and its ancestors n times), so
$= E(\mu(Q_0)^2) \sum_{n=0}^{\infty} b^{\alpha n} (p b^d)^{-2n} p^n = E(\mu(Q_0)^2) \sum_{n=0}^{\infty} b^{n(\alpha - d - \log_b p)}$.
Converges for $\alpha < d + \log_b p$! So $I_\alpha(\mu)$ is a.s. finite.
Thus, if $\mu(C) > 0$, μ gives a measure on C with $I_\alpha(\mu) < \infty$. By lemma, $\{\mu(C)>0\}$ is the same, a.s., as $\{C \ne \emptyset\}$.

Pf of Lemma.
Let $K = E((Z_1 - m)^2) < \infty$ and
Then $E(|Z_{n+1} - m Z_n|^2) = \sum_{k=0}^{b^{nd}} E(|Z_{n+1} - m Z_n|^2 \mid Z_n = j) P(Z_n = j) \le$
j independent guys: $E((\sum a_i)^2) = \sum E(a_i^2)$, if $E(a_i) = 0$.
$\sum_{k=0}^{b^n} (j\, E((Z_1 - m)^2)) \cdot P(Z_n = j) = K \sum j P(Z_n = j) = K E(Z_n) = K m^n$.
Then $E((m^{-n} Z_n - m^{-n-1} Z_{n+1})^2) = m^{-2n-2} E(|Z_{n+1} - m Z_n|^2) = K m^{-2} m^{-n}$.
So $\sum E((m^{-n} Z_n - m^{-n-1} Z_{n+1})^2) < \infty$, which means $\exists \lim_{n\to\infty} m^{-n} Z_n$ in L^2.
The same inequality imply a.s. convergence,
since
$E \sum |m^{-n} Z_n - m^{-n-1} Z_{n+1}| = \sum E \le \sum \left(E\left((m^{-n} Z_n - m^{-n-1} Z_{n+1})^2\right)\right)^{1/2} < \infty$.
so $\sum$ converges a.s.
Now, notice that: $\{\omega : C = \emptyset\}$ is the same as $\{\exists n : Z_n = 0\}$.
Since obviously $\{\omega : \mu(C) = 0\} \supset \{\omega : C = \emptyset\}$, we just need to show that the sets have the same probability.
For this, consider
generating polynomial: $f(x) := \sum_{k=0}^{b^d} q_k x^k$
Note: $f(1) = 1$, $f(0) = q_0 = 1 - p$, $f'(1) = \sum k q_k = m$.
Observe also that $f''(x) = \sum k(k-1) q_k x^{k-2} \ge 0$, so f is convex.
Also $f(x) = E(x^{Z_1})$.
But $Z_2 = \sum_{i=1}^{Z_1} \xi_i$, with ξ_i distributed like Z_1 and independent, so
$E(x^{Z_2}) = E(x^{\xi_1 + \dots + \xi_{Z_1}}) = \sum E(x^{\xi_1 + \dots + \xi_k}) P(Z_1 = k) =$
$\sum E(x^{\xi_1}) \dots E(x^{\xi_k}) P(Z_1 = k) = \sum q_k f(x)^k = f(f(x))$.
and, in general, by induction, the generating polynomial of Z_n is
$E(x^{Z_n}) = f^{\circ n}(x)$ - n-th iteration of f!
So $P(Z_n = 0) = f^{\circ n}(0)$.



$f^{\circ n}(0)$ converges to the smallest fixed point of $f(x)$ at $[0,1]$. If $m \le 1$, there is only one such fixed point, 1, so $P(C = \emptyset) = 1$.
For $m > 1$, $\exists$ another fixed point, x_0. So
$P(C = \emptyset) = x_0$.
Now, if $r := P(\mu(C) = 0)$, then
$r = \sum_k P(\mu(C) = 0 \mid Z_1 = k) P(Z_1 = k) = \sum r^k q_k = f(r)$.
We know that $E(\mu(C)) = \lim_{n\to\infty} E(m^{-n} Z_n) = 1$, so $r < 1$,
thus $r = x_0$. ∎